Fibrations I

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1 Fibrations

In the exercises we used the extension problem to motivate the study of cofibrations. The idea was to allow for homotopy-theoretic methods to be introduced to an otherwise very rigid problem. The dual notion is the *lifting problem*. Here $p : E \to B$ is a fixed map and we would like to known when a given map $f : X \to B$ lifts through p to a map into E

$$\begin{tikzcd} & E \arrow[d, "p"] \\ X \arrow[ur, dashed] \arrow[r, "f"] & B. \end{tikzcd} \tag{1.1}$$

Asking for the lift to make the diagram to commute strictly is neither useful nor necessary from our point of view. Rather it is more natural for us to ask that the lift exist up to homotopy. In this lecture we work in the unpointed category and obtain the correct conditions on the map p by formally dualising the conditions for a map to be a cofibration.

Definition 1 *A map $p : E \to B$ is said to have the **homotopy lifting property** (HLP) with respect to a space X if for each pair of a map $f : X \to E$, and a homotopy $H : X \times I \to B$ starting at $H_0 = pf$, there exists a homotopy $\widetilde{H} : X \times I \to E$ such that ,*

1) $\widetilde{H}_0 = f$

2) $p\widetilde{H} = H$.

*The map p is said to be a (Hurewicz) **fibration** if it has the homotopy lifting property with respect to all spaces.* $\square$

Since a diagram is often easier to digest, here is the definition exactly as stated above

$$
\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "in_0"'] & E \arrow[d, "p"] \\
X \times I \arrow[r, "H"] \arrow[ur, dashed, "\widetilde{H}"] & B
\end{tikzcd}
\tag{1.2}
$$

and also in its equivalent adjoint formulation

$$
\begin{tikzcd}
X \arrow[dr, dashed, "\widetilde{H}"] \arrow[drr, bend left, "H"] \arrow[ddr, bend right, "f"'] & & \\
& E^I \arrow[r, "p_*"] \arrow[d, "e_0"] & B^I \arrow[d, "e_0"] \\
& E \arrow[r, "p"] & B.
\end{tikzcd}
\tag{1.3}
$$

The assertion that p is a fibration is the statement that the square in the second diagram is a *weak pullback*. i.e. it satisfies the existence but not uniqueness property of a pullback. This diagram will eventually lead to a characterisation of fibrations in terms of a universal example. For the moment we prefer to continue with the general discussion.

The reference in the definition to the Polish mathematician Witold Hurewicz stems from his pioneering work [1]. We have included it parenthetically due to the fact that there are other types of fibrations considered in the literature. For instance *Serre* fibrations are the maps which have the homotopy lifting property with respect to all CW complexes. These other types of fibrations will not be discussed here, and for us the word fibration will always mean 'Hurewicz fibration'.

Example 1.1

1) For any space X, the unique map $X \to *$ is a fibration.

2) A homeomorphism is a fibration.

3) For any pair of spaces X, Y, the projection $pr_X : X \times Y \to X$ is a fibration. A fibration of this form is said to be **trivial**.

4) Less obviously, we will show later that the exponential map $\exp : \mathbb{R} \to S^1$ is a fibration. In particular not every fibration is trivial.

5) Set $E = I \times \{0\} \cup \{0\} \times I$ and let $p : E \to I$ be the projection onto the first factor. This map is not a fibration, since there is no lift of the identity starting at $(0, 1)$. $\square$

Lemma 1.1 *If $p : E \to B$ is a fibration, then its image is a union of path components of B.*

Proof If $e \in E$ and $l \in B^I$ is a path starting at $p(e)$, then the HLP grants a path $\widetilde{l} : I \to E$ such that $\widetilde{l}(0) = e$ and $p\widetilde{l} = l$. In particular, any point of B connected by a path to a point in $p(E)$ has a preimage in E. ∎

Note that we are not asserting that a fibration need be surjective. This point is often misrepresented in the literature.

Example 1.2

1) For any space X, the unique map $\emptyset \to X$ is a fibration.

2) The inclusion $X \hookrightarrow X_+$ is a fibration. $\Box$

However, Lemma 1.1 does state that a fibration $p : E \to B$ surjects onto each path component which intersects its image non-trivially. Since a consequence of Proposition 1.2 below is that the restriction of p to any path component of B is again a fibration, we see, especially in the case in which B is locally path-connected, that there is often little harm in assuming that p is surjective.

We use this observation mainly as motivation to introduce some standard terminology. Given $p : E \to B$ we call E the **total space** of the fibration and B the **base space**. Given a point $b \in B$ we call $E_b = p^{-1}(b) \subseteq E$ the **fibre** over b. In the case that B is equipped with a basepoint $*$, we write $F = E_*$, and call it the **typical fibre** of p. The relation between these objects is clarified below. We intuitively think of E as being 'larger' than B, essentially an amalgam of B and the typical fibre F (compare the two fibrations $S^1 \times \mathbb{Z} \to S^1$ and $\mathbb{R} \to S^1$).

Proposition 1.2 *The following statements hold.*

1) If $p : E \to B$ and $q : \widetilde{E} \to E$ are fibrations, then the composite $pq : \widetilde{E} \to B$ is a fibration.

2) If $p_i : E_i \to B_i$, $i = 1, 2$, are fibrations, then the product $p_1 \times p_2 : E_1 \times E_2 \to B_1 \times B_2$ is a fibration.

3) If

$$\begin{tikzcd} P \arrow[r] \arrow[d, "q"'] \arrow[dr, phantom, "\ulcorner", very near start] & E \arrow[d, "p"] \\ A \arrow[r] & B \end{tikzcd} \tag{1.4}$$

is a pullback square and $p : E \to B$ is a fibration, then $q : P \to A$ is a fibration.

Proof 1) Suppose given the solid part of the following diagram

$$\begin{tikzcd} X \arrow[dr, dashed, "(2)"'] \arrow[drr, dashed, "(1)"] \arrow[drrr, bend left, "H"] \arrow[ddr, bend right, "f"'] & & & \\ & \widetilde{E}^I \arrow[r] \arrow[d, "e_0"] & E^I \arrow[r] \arrow[d, "e_0"] & B^I \arrow[d, "e_0"] \\ & \widetilde{E} \arrow[r, "q"] & E \arrow[r, "p"] & B. \end{tikzcd} \tag{1.5}$$

The pair of qf and H pose a homotopy lifting problem for p, and since p is a fibration, the dotted arrow labeled (1) can be filled in so as to make the right-hand side of the diagram

commute. Subsequently, the pair of f and (1) pose a lifting problem for q, which can be solved to yield the map labelled (2), and this map also solves the original problem.
2) This is clear since homotopies of products are products of homotopies.
3) Assume given a lifting problem in the form of the solid part of the next diagram

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "in_0"'] & P \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & E \arrow[d, "p"] \\
X \times I \arrow[r, "H"'] \arrow[ur, dashed, "\widetilde{H}"] & A \arrow[r] \arrow[ur, dashed] & B.
\end{tikzcd} \tag{1.6}$$

The right-most dotted arrow can be completed since p has the HLP by assumption. Then the pair of this new map and H specify a unique map $\widetilde{H} : X \times I \to P$ into the pullback space. Checking directly using the uniqueness of the maps granted by the pullback we see that $\widetilde{H}_0 = f$. $\blacksquare$

A consequence of part 3) of the proposition is that if $A \subseteq B$, then the restriction of a fibration $p : E \to B$ to A is a fibration which we will denote

$$p_A = p| : E_A = p^{-1}(A) \to B. \tag{1.7}$$

For a general map $f : X \to B$, we'll prefer to write

$$f^*E = X \times_B E \to X \tag{1.8}$$

for the pullback fibration.

To end this section we'll leave the reader with some simple observations.

Example 1.3

1) Let $p : E \to B$ be a surjective fibration. Assume that $f : X \to B$ is inessential. Then f lifts to E. In particular, if B is contractible then p admits a section. More generally, if $i : A \subseteq B$ is a subspace inclusion and $i \simeq *$, then there is a lift $\widetilde{i} : A \to E$. This is equivalently a section of $E_A \to A$, even though A itself may not be contractible.

2) Assume that $p : E \to B$ is a fibration and $p \simeq q$. Then there is a map $\theta : E \to E$ such that $p\theta = q$.

3) Assume that $p : E \to B$ is a surjective fibration over a nonempty space B, and that for some $b \in B$, the fibre $p^{-1}(b)$ is path-connected. Then E is path-connected if and only if B is path-connected. $\square$

Example 1.4 Let $p : E \to B$ be a fibration with typical fibre F. Then

$$cat(E) \leq (cat(B) + 1)(cat(F) + 1) - 1. \tag{1.9}$$

Proof Fix a categorical cover $U_1, \ldots, U_{m+1}$ of B and a categorical cover $V_1, \ldots, V_{n+1}$ of F. For each $i = 1, \ldots, m+1$ let J^i be a null homotopy of the inclusion $U_i \hookrightarrow B$ and consider the diagram

$$\begin{tikzcd} E_{U_i} \arrow[rr] \arrow[d, "in_0"'] & & E \arrow[d, "p"] \\ E_{U_i} \times I \arrow[r] \arrow[urr, dashed, "K^i"] & U_i \times I \arrow[r, "J^i"] & B \end{tikzcd} \tag{1.10}$$

where $E_{U_i} = p^{-1}(U_i)$. By applying the HLP we get the map K^i, and setting $k_i = K^i_1$ we get $k_i(E_{U_i}) \subseteq F$. Next, for each $j = 1, \ldots, n+1$, put

$$W_{ij} = k_i^{-1}(V_j) \tag{1.11}$$

to get an open cover $W_{i1}, \ldots, W_{i(n+1)}$ of E_{U_i}. Since the E_{U_i} cover E, so do the sets W_{ij}. We show below that these $(m+1)(n+1)$ sets constitute a categorical cover of E.

Now, for each $j = 1, \ldots, n+1$ choose a null homotopy L^j of the inclusion $V_j \hookrightarrow F$. Then we get a null homotopy M^{ij} of the inclusion $W_{ij} \hookrightarrow E$ by pasting together the homotopies in the following diagram

$$\begin{tikzcd} & & * \arrow[d] \\ & V_j \arrow[ur] \arrow[r] \arrow[ur, phantom, shift left=2, "\overset{L^j}{\Leftarrow}"] & F \arrow[d] \\ W_{ij} \arrow[ur, "k_i"] \arrow[r] & E_i \arrow[r, hook] \arrow[ur, phantom, "\overset{K^i}{\Leftarrow}"] & E. \end{tikzcd} \tag{1.12}$$

Explicitly

$$M^{ij}(e,t) = \begin{cases} K^i(e, 2t) \\ L^j(k_i(e), 2t-1) \end{cases} \qquad e \in W_{ij}, t \in I. \quad \blacksquare \tag{1.13}$$

2 The Mapping Path Space

Definition 2 *Given a map $f : X \to Y$ we define its* ***mapping path space*** W_f *to be the pullback in the square*

$$\begin{tikzcd} W_f \arrow[r, "q_f"] \arrow[d, "\pi_f"'] \arrow[dr, phantom, "\ulcorner", very near start] & Y^I \arrow[d, "e_0"] \\ X \arrow[r, "f"] & Y. \end{tikzcd} \tag{2.1}$$

In particular

$$W_f \cong \{(x, l) \in X \times Y^I \mid f(x) = l(0)\} \tag{2.2}$$

and the maps $\pi_f : W_f \to X$ and $q_f : W_f \to Y^I$ which appear in (2.1) *will be our favoured notation for the the projections onto the first and second factors, respectively.* $\square$

Now, with $f : X \to Y$ fixed we get a map $\widetilde{f} : X^I \to W_f$ as that induced by the diagram

$$\begin{tikzcd}
X^I \arrow[dr, dashed, "\widetilde{f}"] \arrow[drr, bend left, "f_*"] \arrow[ddr, bend right, "f"'] & & \\
& W_f \arrow[r, "q_f"] \arrow[d, "\pi_f"'] \arrow[dr, phantom, "\ulcorner", very near start] & Y^I \arrow[d, "e_0"] \\
& X \arrow[r, "f"] & Y.
\end{tikzcd} \tag{2.3}$$

Proposition 2.1 *A map $f : X \to Y$ is a fibration if and only if the associated map $\widetilde{f} : X^I \to W_f$ has a right inverse $\lambda : W_f \to X^I$.*

Proof The pullback square defining W_f specifies a homotopy lifting problem for f. Thus if f is a fibration, then there exists a solution λ which makes the next diagram commute

$$\begin{tikzcd}
W_f \arrow[dr, dashed, "\lambda"] \arrow[drr, bend left, "q_f"] \arrow[ddr, bend right, "\pi_f"'] & & \\
& X^I \arrow[r, "f_*"] \arrow[d, "e_0"] & Y^I \arrow[d, "e_0"] \\
& X \arrow[r, "f"] & Y.
\end{tikzcd} \tag{2.4}$$

We check that

$$\pi_f(\widetilde{f}\lambda) = e_0\lambda = \pi_f, \qquad q_f(\widetilde{f}\lambda) = f_*\lambda = q_f \tag{2.5}$$

which implies that $\widetilde{f}\lambda = id_{W_f}$. This gives us the forwards implication.

Conversely, assume that λ is a right inverse to the map $\widetilde{f}$. Note that this implies the equations

$$e_0\lambda = (\pi_f\widetilde{f})\lambda = \pi_f, \qquad f_*\lambda = (q_f\widetilde{f})\lambda = q_f. \tag{2.6}$$

To complete the proof we need to show that f has the homotopy lifting property with respect to a given space A. So assume given the lifting problem on the left below

$$\begin{tikzcd}
A \arrow[dr, dashed] \arrow[drr, bend left, "H"] \arrow[ddr, bend right, "\alpha"'] & & \\
& X^I \arrow[r, "f_*"] \arrow[d, "e_0"] & Y^I \arrow[d, "e_0"] \\
& X \arrow[r, "f"] & Y
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
A \arrow[dr, "\theta"] \arrow[drr, bend left, "H"] \arrow[ddr, bend right, "\alpha"'] & & \\
& W_f \arrow[r, "q_f"] \arrow[d, "\pi_f"'] \arrow[dr, phantom, "\ulcorner", very near start] & Y^I \arrow[d, "e_0"] \\
& X \arrow[r, "f"] & Y.
\end{tikzcd} \tag{2.7}$$

Since the square on the right is a pullback, the maps α, H determine the indicated map $\theta = \theta(\alpha, H) : A \to W_f$. Define

$$\widetilde{H} = \lambda\theta : A \to X^I. \tag{2.8}$$

Then

$$e_0\widetilde{H} = e_0(\lambda\theta) = \pi_f\widetilde{H} = \alpha, \qquad f_*\widetilde{H} = f_*(\lambda\theta) = q_f\theta = H. \tag{2.9}$$

In particular $\widetilde{H}$ solves the lifting problem on the left of (2.7). ∎

This proposition supplies the universal example we alluded to in the opening section of this lecture. A consequence is that $f : X \to Y$ has the HLP with respect to all spaces if and only if it has the HLP with respect to W_f. The parallels with the theory of cofibrations are now manifest: W_f is dual to the mapping cylinder M_f, and the lift $\lambda : W_f \to X^I$ is dual to the retraction $X \times I \to M_f$.

Definition 3 *Let $p : E \to B$ be a fibration and $\widetilde{p} : E^I \to W_p$ the map constructed in* (2.3). *Any choice of section $\lambda : W_p \to E^I$ of this map is said to be a **lifting function** for p.* □

Example 2.1 In this example we revisit Proposition 1.2 and as an exercise rederive the results there in terms of lifting functions.

1) Let $f : X \to Y$ and $g : Y \to Z$ be fibrations, and choose for them lifting functions $\lambda_f : W_f \to X^I$ and $\lambda_g : W_g \to Y^I$. There is a canonical sequence of maps

$$W_f \xrightarrow{1\times g_*} W_{gf} \xrightarrow{f\times 1} W_g \tag{2.10}$$

and if $(x, k) \in W_{gf}$, then

$$\lambda_g(f(x), k)(0) = \pi_g(f(x), k) = f(x). \tag{2.11}$$

In particular $(x, \lambda_g(f(x), k)) \in W_f$. Define $\lambda_{gf} : W_{gf} \to Y^I$ to be the map

$$\lambda_{gf}(x, k) = \lambda_f(x, \lambda_g(f(x), k)). \tag{2.12}$$

We check easily that this is a lifting function for gf, and conclude that this composition is a fibration.

2) Let $p_i : E_i \to B_i$, $i = 1, 2$, be fibrations. Then there is a canonical homeomorphism $W_{p_1\times p_2} \cong W_{p_1} \times W_{p_2}$, and choices of lifting functions λ_1 for p_1 and λ_2 for p_2 induce a lifting function for $p_1 \times p_2$. Thus $p_1 \times p_2 : E_1 \times E_2 \to B_1 \times B_2$ is a fibration.

3) Let $p : E \to B$ be a fibration. Given a map $f : X \to B$ let $q : X \times_B E \to X$ be its pullback. Then there is a commutative cube in which the left-hand, right-hand and bottom faces are are pullbacks

$$\begin{tikzcd}
W_q \arrow[rr] \arrow[dr] \arrow[dd] & & W_p \arrow[dr] \arrow[dd] & \\
& X^I \arrow[rr] \arrow[dd] & & B^I \arrow[dd] \\
X \times_B E \arrow[rr] \arrow[dr, "q"'] & & E \arrow[dr, "p"] & \\
& X \arrow[rr, "f"] & & B.
\end{tikzcd} \tag{2.13}$$

Checking the universal property we see that the top face of this cube is also a pullback, so in particular there is a homeomorphism $W_q \cong X^I \times_{B^I} W_p$. This can also be checked directly. Choose a lifting function $\lambda_p : W_p \to B^I$ for p and let λ_q be the composite

$$W_q \cong X^I \times_{B^I} W_p \xrightarrow{1\times\lambda_p} X^I \times_{B^I} E^I \cong (X \times_B E)^I \tag{2.14}$$

to get a lifting function for q.

4) The last example is clarified somewhat by replacing the map f with a subspace inclusion $A \hookrightarrow B$. In this case $E_A \subseteq E$, and the map $W_{p_A} \to W_p$ is injective. If $\lambda : W_p \to B^I$ is a lifting function for p, then $\lambda(W_{p_A}) \subseteq A^I$, so we get a lifting function for p_A simply by restriction.

5) Finally we consider the trivial fibration $pr_X : X \times Y \to X$. There is a homeomorphism $W_{pr_X} \cong X^I \times Y$, so a lifting function for this fibration is furnished by any choice of map $s : Y \to Y^I$ satisfying $s(y)(0) = y$. In particular the map sending each point to the constant path at that point will do. $\square$

Lest these examples lead the reader to believe that lifting functions are only useful in proving technical results, here is another example to highlight the power of having a universal example.

Example 2.2 Let $p : E \to B$ be a map between metric spaces E, B. Assume that p has the homotopy lifting property with respect to all metric spaces. Then p is a fibration. We can see this by first observing that the compact-open topology on B^I agrees with the topology of uniform convergence, since B is metric, and so is itself metrisable, since I is compact. It follows that the subspace $W_p \subseteq E \times B^I$ is also metrisable, and so we can thus find a lifting function for p by assumption. $\square$

Example 2.3 *Let $p : \widetilde{X} \to X$ be a map with unique path lifting.*

$$\begin{tikzcd} \widetilde{X} \arrow[r, "q_f"] \arrow[d, "\pi_f"'] \arrow[dr, phantom, "\ulcorner", very near start] & X^I \arrow[d, "e_0"] \\ \widetilde{X} \arrow[r, "p"] & X. \end{tikzcd} \tag{2.15}$$

Finally we'll end this section with a technical result which is the dual of Proposition 1.2 from the exercises. Note, however, that the symmetry is not elegant, and we must enforce additional assumptions on the base space in order to obtain a dual statement.

Proposition 2.2 *If $p : E \to B$ is a surjective fibration over a locally path-connected space B, then p is a quotient map.*

Proof Form the mapping path space W_p and let $q : W_p \to B$ be the map $(e, l) \mapsto l(1)$. Then the diagram

$$\begin{tikzcd} E^I \arrow[r, "\widetilde{p}"] \arrow[d, "e_0"'] & W_p \arrow[d, "\pi"] \\ E \arrow[r, "p"] & B. \end{tikzcd} \tag{2.16}$$

commutes by construction. The map e_0 is a quotient map since it has a section. Similarly the map $\widetilde{p}$ is a quotient map, since any choice of lifting function λ for p is a section. It follows now from the commutativity of (2.16) that p is a quotient map if and only if π_p is. Thus we will be done if we can show that q is a quotient map.

So suppose that $A \subseteq B$ is a nonempty set such that $q^{-1}(A)$ is open in W_p. Fix a point $b \in A$ and a preimage $e \in p^{-1}(b)$. Also let $c_b \in B^I$ be the constant path at b. Now, if

$U \subseteq B$ is an open neighbourhood of b, then U^I is naturally embedded in B^I which carries the compact-open topology. Since W_p is topologised as a subspace of $E \times B^I$, it is not difficult to see we can choose the neighbourhood U such that $(\{e\} \times U^I) \cap W_p \subseteq q^{-1}(A)$.

Let $U_0 \subseteq U$ be the path-component of b. Note that this set is open in B since B is locally path-connected. We claim that $U_0 \subseteq A$, which, if true, will show that A is open, and so complete the proof. To see the claim let $b' \in U_0$ be any point and choose a path $l \in U^I$ such that $l(0) = b$ and $l(1) = b'$. Then $(e, l) \in (\{e\} \times U)^I \cap W_p$ and $q(e, l) = b'$. Hence $b' \in A$, and we can see the claim is true. $\blacksquare$

Example 2.4

1) Every CW complex is locally path-connected [2]. Every (topological) manifold is locally-path connected. In particular, Pr. 2.2 applies to all fibrations over such spaces.

2) Let $\mathbb{Q}$ be the rationals in its standard topology and Q the rationals in the discrete topology. The identity $Q \to \mathbb{Q}$ is a fibration, but clearly not a quotient map. Of course $\mathbb{Q}$ is not locally path-connected. $\square$

3 Mapping Spaces and Fibrations

In this section we study the interactions between fibrations and function spaces. The main tool for this is the use of adjunction, and it is not surprising to see some local compactness requirements. The main results are Propositions 3.1 and 3.1.

Proposition 3.1 *Let B be a pointed space and $p : E \to B$ be a fibration with typical fibre F. Then for any locally compact space X, the postcomposition map*

$$p_* : C(X, E) \to C(X, B) \tag{3.1}$$

is a fibration with typical fibre $C(X, F)$.

Proof Consider the following two diagrams

$$\begin{tikzcd} K \arrow[r] \arrow[d, "in_0"'] & C(X,E) \arrow[d, "p_*"] \\ K \times I \arrow[r] \arrow[ur, dashed] & C(X,B) \end{tikzcd} \qquad \begin{tikzcd} X \times K \arrow[r] \arrow[d, "in_0"'] & E \arrow[d, "p"] \\ X \times K \times I \arrow[r] \arrow[ur, dashed] & B. \end{tikzcd} \tag{3.2}$$

The left-hand diagram is a lifting problem for p_*. The right-hand diagram is obtained from the left-hand side by adjunction. Since X is locally compact, all the maps in this diagram are continuous, and since p is a fibration, the dotted arrow can be filled in. The adjoint of this filler is a map $K \times I \to C(X, E)$ which then solves the lifting problem in the left-hand diagram. $\blacksquare$

Proposition 3.2 *Assume that $j : A \hookrightarrow X$ is a closed cofibration in a locally compact space X. Then for any space Y, the precomposition map*

$$j^* : C(X, Y) \to C(A, Y) \tag{3.3}$$

is a fibration.

Proof Consider the next two diagrams

$$
\begin{tikzcd}
K \arrow[dr, dashed] \arrow[drr, bend left, "H"] \arrow[ddr, bend right, "f"'] & & \\
& C(X,A)^I \arrow[r] \arrow[d, "e_0"] & C(A,Y)^I \arrow[d, "e_0"] \\
& C(X,Y) \arrow[r, "j^*"] & C(A,Y)
\end{tikzcd}
\qquad
\begin{tikzcd}
K \times A \arrow[r, hook, "in_0"] \arrow[d, "1\times j"'] & K \times A \times I \arrow[d, "1\times j\times 1"'] \arrow[ddr, bend left, "H^\flat"] & \\
K \times X \arrow[r, "in_0"] \arrow[drr, bend right, "f^\flat"'] & K \times X \times I \arrow[dr, dashed, "\widetilde{H}"] & \\
& & Y.
\end{tikzcd}
\tag{3.4}
$$

The left-hand diagram represents an assumed lifting problem for j^*. The right-hand diagram is obtained from the left-hand side by adjunction. Since $A \subseteq X$ is closed, it is locally compact in the subspace topology, so all the maps in the right-hand diagram are continuous. Moreover, since j is a cofibration, the dotted arrow marked $\widetilde{H}$ on the right-hand side can be filled in. The adjoint of this is a map $K \to C(X \times I, Y) \cong C(X,Y)^I$ which completes the dotted arrow in the left-hand diagram. ∎

In general it is difficult to identify the fibres of (3.3), although we find an exception to this in the following corollary, where we can even forego the closedness assumption.

Corollary 3.3 *Let X be a pointed space and assume that X is locally compact and that the inclusion $* \hookrightarrow X$ is a cofibration. Then for any pointed space Y, the evaluation map $ev : C(X,Y) \to Y$, $f \mapsto f(*)$, is a fibration with typical fibre $C_*(X,Y)$.* ∎

Example 3.1 Fix a pointed space X. The mapping space $LX = C(S^1, X)$ is called the **free loop space** of X. The map $ev : LX \to X$ is a fibration with typical fibre ΩX. □

As special but important cases of Proposition 3.2 and Corollary 3.3 we have the following.

Corollary 3.4 *Let X be space*

1) *For each $t \in I$, the evaluation map $e_t : X^I \to X$, $l \mapsto l(t)$, is a fibration.*

2) *The start-end evaluation map $e_{0,1} : X^I \to X \times X$, $l \mapsto (l(0), l(1))$, is a fibration.*

3) *If X has a basepoint $*$, then the start point evaluation map $e_1 : PX \to X$ is a fibration with fibre ΩX.*

Proof For the first part we apply Pr. 3.2 to the cofibration $in_t : * \hookrightarrow I$. For the second part we use the cofibration $\partial I \hookrightarrow I$. The last part comes from the pullback diagram

$$
\begin{tikzcd}
PX \arrow[r] \arrow[d, "e_0"'] \arrow[dr, phantom, "\lrcorner", very near start] & X^I \arrow[d, "e_{0,1}"] \\
X \arrow[r, "(id_X, *)"] & X \times X.
\end{tikzcd}
\tag{3.5}
$$

∎

Having established the fibration 3) above we can now prove the following, which was claimed without proof in an earlier lecture.

Corollary 3.5 *Every nonempty space X which is both path-connected and locally path-connected is the quotient of a contractible space.*

Proof Choose a basepoint $* \in X$. Then $e_0 : PX \to X$ is the required quotient. The map e_0 is a fibration by Corolalry 3.4, and so a quotient by Proposition 2.2. The total space PX was shown to be contractible in Lemma 1.5 of the lecture *pointed homotopy.* ∎

The last result of this section is the pointed analogue of Proposition 3.2.

Proposition 3.6 *Let $j : A \hookrightarrow X$ be a closed cofibration. Assume that X, A are based spaces and the map j preserves the basepoint. Then for any pointed space Y, the precomposition map*

$$j^* : C_*(X, Y) \to C_*(A, Y) \tag{3.6}$$

is a fibration with fibre $C_*(X/A, Y)$.

Proof We check that the square

$$\begin{tikzcd} C_*(X,Y) \arrow[r] \arrow[d, "j^*"'] \arrow[dr, phantom, "\ulcorner", very near start] & C(X,Y) \arrow[d, "j^*"] \\ C_*(A,Y) \arrow[r] & C(A,Y) \end{tikzcd} \tag{3.7}$$

is a pullback. The right-hand map is a fibration by Pr. 3.2, and thus the left-hand map is a fibration by Pr. 1.2. It is easy to identify the fibre. ∎

4 Exercises

General Theory

1) Let $p : E \to B$ be a fibration over a pointed space B. Assume that B is simply-connected and E is path-connected. Show that $F = p^{-1}(*)$ is connetced.

2) Let $p : E \to B$ be a fibration over a path-connected space B. Let F be the fibre over a chosen basepoint $* \in B$ and assume that the inclusion $i : F \hookrightarrow E$ is null homotopic. Show that $cat(E) \leq cat(B)$.

Regular Fibrations A fibration allows for solutions to a given homotopy lifting problems to be found. Often it is desirable to have some control over these solutions. One of the mildest assumptions we can make in this regard is that of *regularity*, which is the requirement that constant homotopies may be lifted to constant homotopies. We make this precise as follows.

Definition 4 *A fibration $p : E \to B$ is said to be **regular** if it admits a lifting function $\lambda : W_p \to E^I$ with the property that $\lambda(e, l) \in E^I$ is a constant path whenever $l \in B^I$ is.* □

1) Let $p : E \to B$ be a fibration. Show that p is regular if and only if any homotopy lifting problem

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "in_0"'] & E \arrow[d, "p"] \\ X \times I \arrow[r, "H"] \arrow[ur, dashed] & B \times I \end{tikzcd} \tag{4.1}$$

admits a solution $\widetilde{H} : X \times I \to E$ with the property that if $x_0 \in X$ is such that $t \mapsto H_t(x_0)$ is a constant path, then so is $t \mapsto \widetilde{H}_t(x_0)$.

2) Let $pr_1 : B \times F \to B$ be the projection where F is nonempty. Show that pr_1 is a regular fibration.

3) Let $p : E \to B$ is a regular fibration and $f : X \to B$ is a map. Show that the pullback $f^*E \to X$ is a regular fibration.

4) Let B be a space. Assume that there is a map $\phi : B^I \to I$ such that $\phi(l) = 0$ if and only if l is constant. Show that any fibration $p : E \to B$ is regular.

5) Let $p : E \to B$ be a fibration over a metrisable space B. Show that p is regular.

6) Let $\mathcal{S}$ be the Sierpinski space. Show that the evaluation fibration $e_0 : \mathcal{S}^I \to \mathcal{S}$ is not regular.

7) Let $p : E \to B$ be a regular fibration. Assume that $A \subseteq B$ is a strong deformation retract. Show that $p^{-1}(A) \subseteq E$ is a strong deformation retract.

References

[1] W. Hurewicz, *On the Concept of Fiber Space*, Proc. Nat. Acad. Sci. USA **41** (1955), 956-961.

[2] A. Lundell, S. Weingram, *The Topology of CW Complexes*, Van Nostran, (1969).